Nonstationary analogues of the Herglotz representation theorem for unbounded operators

Alpay, D.; Bolotnikov, V.; Dijksma, A.; Freydin, B.

Published in:
Archiv der mathematik

Publication date:
2002

Arch. Math. 78 (2002) 465–474
0003-889X/02/060465-10 $ 3.50/0

Archiv der Mathematik

Nonstationary analogues of the Herglotz representation theorem for unbounded operators

By

D. Alpay, V. Bolotnikov, A. Dijksma[1]) and B. Freydin

Abstract. We establish a Herglotz type representation theorem for (possibly unbounded) upper triangular operators with positive real part.

1. Introduction. Herglotz's theorem asserts that every function ϕ which is analytic in the open unit disk $\mathbb{D}$ with a positive real part there (i.e. a Carathéodory function) admits an integral representation of the form

$$\phi(z) = \operatorname{Im} \phi(0) + \int_0^{2\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}}\, d\sigma(t) \tag{1.1}$$

for some nonnegative finite measure $d\sigma$; see e.g. [6, Theorem 4.5]. Formula (1.1) can be rewritten as

$$\phi(z) = \operatorname{Im} \phi(0) + C\,(I + zU)\,(I - zU)^{-1}\, C^*, \tag{1.2}$$

where $C : L_2(d\sigma) \to \mathbb{C}$ is defined by $Cf = \int_0^{2\pi} f(t)\, d\sigma(t)$ and U is the operator of multiplication by e^{-it} in the space $L_2(d\sigma)$, which is obviously unitary.

The purpose of this paper is to establish an analogue of formula (1.2) in the setting where bounded analytic functions are replaced by upper triangular operators (see Definition 2.1). For a related work, see [1]. This framework corresponds to nonstationary discrete time systems; see e.g. [7]. We send the reader also to the works [3], [10], [8] and [11] for more references and information on the theory of nonstationary systems. We expect our results to have implications in this theory. The main result of the paper is Theorem 4.1, in which we present a generalization to the case of unbounded operators of the results of [4]. Since Carathéodory functions need not be bounded (for instance, take $\phi(z) = \frac{1-z}{1+z}$), there is a need to consider unbounded upper triangular operators with a positive real part. In the passage from the bounded to the unbounded case a number of difficulties arise (see Example 2.3). Two key ingredients in our analysis are: the definition of an upper triangular unbounded operator with

Mathematics Subject Classification (2000): 93Cxx, 30D50, 47N70.

[1]) A. Dijksma gratefully acknowledges support through a Harry T. Dozor fellowship at the Ben-Gurion University of the Negev, Beer-Sheva, Israel, in the year 1999.

positive real part (see Definition 2.4) and the use of the "Zadeh" extension of a sequence of diagonal operators (see Definition 2.2).

2. Carathéodory operators. We let $\ell_2(\mathbb{Z})$ denote the Hilbert space of square summable double–infinite sequences indexed by the integers $\mathbb{Z}$. Let $\{e_j\}$ be the standard basis of $\ell_2(\mathbb{Z})$ and denote by Z the bilateral shift on $\ell_2(\mathbb{Z})$: $Ze_n = e_{n+1}$.

Definition 2.1. A bounded operator T from $\ell_2(\mathbb{Z})$ into itself is upper triangular (lower triangular, diagonal) if its matrix representation with respect to the standard basis of $\ell_2(\mathbb{Z})$ is upper triangular (lower triangular or diagonal, respectively).

We will denote by $\mathscr{U}$ and $\mathscr{D}$ the set of bounded upper triangular and diagonal operators. If $T \in \mathscr{U}$, then there is a sequence of diagonal operators $T_0, T_1, \dots$ such that for all $N \in \mathbb{N}$

$$T - \sum_0^N T_j Z^j \in Z^{N+1}\mathscr{U}. \tag{2.1}$$

The T_j are uniformly bounded in norm (in fact, if T has the matrix representation $T = \big(t_{\ell,k}\big)$, then $\sup_j \|T_j\| = \sup_{\ell,k} |t_{\ell,k}| \leqq \|T\|$), hence the series

$$T(r) = \sum_0^\infty r^j T_j Z^j \tag{2.2}$$

converges in the operator norm for all $r \in [0, 1)$ and is called the Zadeh extension of T. The Zadeh extension has a number of nice properties. In particular,

$$(T_1T_2)(r) = T_1(r)T_2(r), \quad r \in [0, 1), \tag{2.3}$$

for any two upper triangular bounded operators T_1 and T_2.

Definition 2.2. The Zadeh extension of a sequence of bounded diagonal operators $\Psi_0, \Psi_1, \dots$ is the operator–valued function

$$\Psi(r) = \sum_0^\infty \Psi_n Z^n r^n. \tag{2.4}$$

If the sequence Ψ_j defines a bounded upper triangular operator Ψ (as in (2.1)) then $\Psi(r)$ in (2.4) is defined for all $r \in [0, 1)$ and coincides with $\Psi(r)$ in (2.2). In general, $\Psi(r)$ need not converge in all of $[0, 1)$ and in particular it is not the Zadeh extension of a bounded operator. Note also that $\Psi(r)$ in (2.4) exists and defines a bounded operator for $r \in [0, 1)$ under much weaker conditions. For instance, if the diagonal operators Ψ_j are polynomially bounded ($\|\Psi_j\| \leqq Cj^n$ for a fixed n), then the series in (2.4) converges in the operator norm for every $r \in [0, 1)$ but it need not be the Zadeh extension of an operator in $\mathscr{U}$.

The Zadeh extension of elements of $\mathscr{U}$ was used in [9] to solve some nonstationary interpolation problems. In the present work we consider it in the setting of unbounded operators. We note that even the definition of an upper triangular unbounded operator is not so clear, as is illustrated by the following example.

Example 2.3. If $\{e_j\}$ is the standard basis of $\ell_2(\mathbb{Z})$, then in the weak operator topology,

$$\left(\sum_{n=0}^\infty Z^n\right)(e_j - e_{j-1}) = -\left(\sum_{n=1}^\infty Z^{-n}\right)(e_j - e_{j-1}),$$

where the operator on the left is unbounded and "upper triangular" while the operator on the right is unbounded and "lower triangular".

These operators are the analogues of the two Laurent expansions of the function $\frac{1}{1-z}$ centered at the origin. We will call a densely defined unbounded operator T *upper triangular* (with respect to the connected open set $\mathscr{O}$ of its resolvent set) if the operator $(\lambda I - T)$ is upper triangular for some (and hence all) $\lambda \in \mathscr{O}$.

Definition 2.4. An operator $\Phi : \ell_2(\mathbb{Z}) \to \ell_2(\mathbb{Z})$ is called a Carathéodory operator if

1. dom $\Phi \cap$ dom Φ^* is dense.
2. $\langle (\Phi + \Phi^*)x, x\rangle \geqq 0, \quad x \in \operatorname{dom} \Phi \cap \operatorname{dom} \Phi^*$.
3. The operator $(I + \Phi)^{-1}$ is upper triangular with respect to the standard basis of $\ell_2(\mathbb{Z})$ and $(I + \Phi)^{-1}(0)$ boundedly invertible.

Remark 2.5. That $I + \Phi$ in Definition 2.4 is boundedly invertible can be seen as follows: For every $x \in \operatorname{dom} \Phi \cap \operatorname{dom} \Phi^*$,

$$\begin{aligned}\|(\Phi + I)\, x\|^2_{\ell^2(\mathbb{Z})} &= \|\Phi x\|^2_{\ell^2(\mathbb{Z})} + \langle \Phi x, x\rangle_{\ell^2(\mathbb{Z})} + \langle x, \Phi x\rangle_{\ell^2(\mathbb{Z})} + \|x\|^2_{\ell^2(\mathbb{Z})}\\ &= \|\Phi x\|^2_{\ell^2(\mathbb{Z})} + \langle (\Phi + \Phi^*)\, x, x\rangle_{\ell^2(\mathbb{Z})} + \|x\|^2_{\ell^2(\mathbb{Z})}\\ &\geqq \|x\|^2_{\ell^2(\mathbb{Z})}\,.\end{aligned}$$

Thus,

$$\|\,(\Phi + I)\, x\|_{\ell^2(\mathbb{Z})} \geqq \|x\|_{\ell^2(\mathbb{Z})}$$

and quite similarly,

$$\|\,(\Phi^* + I)\, x\|_{\ell^2(\mathbb{Z})} \geqq \|x\|_{\ell^2(\mathbb{Z})}.$$

Then, by the open mapping principle, the operator $I + \Phi$ is boundedly invertible from $\ell^2(\mathbb{Z})$ into itself.

Let Φ be a Carathéodory operator and let S be the upper triangular operator defined by $S = -I + 2\,(I + \Phi)^{-1}$. Then

$$I - S^*S = 2\,(I + \Phi)^{-1} + 2\,(I + \Phi)^{-*} - 4\,(I + \Phi)^{-*}\,(I + \Phi)^{-1}\,. \tag{2.5}$$

For $x = (\Phi + I)y,\ y \in \operatorname{dom} \Phi$, we have by (2.5)

$$\begin{aligned}\langle (I - S^*S)x,\ x\rangle_{\ell_2(\mathbb{Z})} &= \langle (I - S^*S)(\Phi + I)y,\ (\Phi + I)y\rangle_{\ell_2(\mathbb{Z})}\\ &= 2\langle (\Phi + I)y,\ y\rangle_{\ell_2(\mathbb{Z})} + 2\langle y,\ (\Phi + I)y\rangle_{\ell_2(\mathbb{Z})} - 4\langle y,\ y\rangle_{\ell_2(\mathbb{Z})}\\ &= 2\langle \Phi y,\ y\rangle_{\ell_2(\mathbb{Z})} + 2\langle y,\ \Phi y\rangle_{\ell_2(\mathbb{Z})}\\ &= 2\langle (\Phi + \Phi^*)y,\ y\rangle_{\ell_2(\mathbb{Z})} \geqq 0.\end{aligned}$$

From $(\operatorname{ran}\,(\Phi + I))^\perp = \ker\,((\Phi + I)^*) = \{0\}$ it follows that ran $(\Phi + I)$ is dense and the above inequality implies that S is a contraction.

Theorem 2.6. *Let Φ be a Carathéodory operator and assume that the main diagonal of the bounded operator $(I + \Phi)^{-1}$ is invertible. Then there exists a sequence of diagonal operators $\widehat{\Phi}_0,\ \widehat{\Phi}_1, \ldots$ such that its Zadeh extension $\Psi(r)$ satisfies*

$$\left((I + \Phi)^{-1}\right)(r) = (I + \Psi(r))^{-1}, \quad r \in [0, \epsilon) \tag{2.6}$$

for some $\epsilon > 0$.

Proof. Since $(I+\Phi)^{-1} \in \mathscr{U}$, we can write

$$(I+\Phi)^{-1} = \sum_0^\infty D_j Z^j, \quad D_j \in \mathscr{D}$$

(in the sense of (2.1)). The main diagonal D_0 is bounded and, by hypothesis, has a bounded inverse. Thus,

$$(I+\Phi)^{-1}(r) = \sum_0^\infty r^j D_j Z^j = D_0\big(I + D_0^{-1} \sum_1^\infty r^j D_j Z^j\big).$$

For r small enough, the upper triangular operator $D_0^{-1}(\sum_{j=1}^\infty r^j D_j Z^j)$ has norm strictly less than one. The operator $(I+\Phi)^{-1}(r)$ is invertible in $\mathscr{U}$ for such r. Writing its inverse in the form $I + \sum_{j=0}^\infty \Phi_j(r) Z^j$, it is easily checked by induction that the $\Phi_j(r)$ are of the form $\Phi_j(r) = r^j \widehat{\Phi}_j$. $\square$

We will call $\widehat{\Phi}_0, \widehat{\Phi}_1, \ldots$ *the associated sequence* to the Carathéodory operator Φ.

If Φ in Theorem 2.6 is bounded, then it follows from (2.3) applied to $I = (I+\Phi)(I+\Phi)^{-1}$ that $\Phi(r) = \Psi(r)$. For an unbounded Carathéodory operator Φ the relationship between $\Psi(r)$ and Φ is not so clear ($\Psi(r)$ need not be defined in all of $[0,1)$). However, under an appropriate hypothesis, we have $\text{w-}\lim_{r\to 1} \Psi(r) = \Phi$ as shown in the next theorem.

Theorem 2.7. *Let Φ be a Carathéodory operator such that the main diagonal of $(I+\Phi)^{-1}$ is equal to $\frac{1}{2}I$ and let $S = -I + 2(I+\Phi)^{-1}$. Assume that $\sum_1^\infty n\|S_n\| < \infty$, where the S_n are the diagonal operators defining S. Then, on the range* ran $(I+S)$ *the operator valued function $\Psi(r)$ tends to Φ in the weak operator topology as r tends to* 1.

Proof. By definition of the Zadeh extension and from (2.6) we have

$$S(r) = -I + 2(I+\Phi)^{-1}(r) = -I + 2(I+\Psi(r))^{-1}. \tag{2.7}$$

Since

$$\tfrac{1}{2}(\Phi - \Psi(r)) = (I+S)^{-1} - (I+S(r))^{-1},$$

it is enough to prove that $(I+S(r))^{-1}$ tends to $(I+S)^{-1}$. Next, for every $y \in \mathscr{D}$

$$((I+S(r))^{-1} - (I+S)^{-1})y = (I+S(r))^{-1}(S - S(r))x.$$

By (2.7), $S(0) = 0$, i.e.,

$$S(r) = \sum_{n=1}^\infty S_n Z^n r^n \tag{2.8}$$

and, by the nonstationary analogue of Schwartz lemma (see [9, Theorem 5.5 p. 136]), we have $\|(I+S(r))^{-1}\| \leqq (1-r)^{-1}$. The assumptions of the theorem imply

$$\|S - S(r)\| = \left\| \sum_{n=1}^\infty S_n Z^n (1-r^n) \right\| \leqq (1-r)\left(\sum_{n=1}^\infty n\,\|S_n\|\right)$$

and thus,

$$\|(I+S(r))^{-1}(S-S(r))\| \leqq C, \quad \text{where} \quad C = \sum_{n=0}^\infty n\|S_n\|. \tag{2.9}$$

This uniform bound implies that $(I+S(r))^{-1}(S-S(r))x$ is weakly convergent to 0 for every x if and only if it is weakly convergent for $x=e_k$, $k\in\mathbb{Z}$. Then to prove the theorem it suffices to show that

$$\text{w-}\lim_{r\to 1}|(I+S(r))^{-1}(S-S(r))e_k=0 \quad \text{for} \quad k\in\mathbb{Z}.$$

For this proof, recall that $S(0)=0$ and write $S=Z\sigma$ where $\sigma\in\mathscr{U}$ has norm less than 1. Then,

$$(I+S(r))^{-1}=1-rZ\sigma(r)+r^2Z\sigma(r)Z\sigma(r)-\cdots$$

and so each coefficient $B_n(r)$ in the expansion

$$(I+S(r))^{-1}(S-S(r))=\sum_{n=1}^{\infty}B_n(r)Z^n$$

admits a bound of the form $\|B_n(r)\|\leqq n(1-r)$ and tends to 0 in norm as r tends to 1. Since $Z^ne_k=e_{k+n}$, it is readily seen that for every $y=\sum_{-\infty}^{\infty}y_ne_n\in\ell_2(\mathbb{Z})$,

$$\begin{aligned}\left\langle\left(\sum_{i=m-k+1}^{\infty}B_i(r)Z^i\right)e_k,\,y\right\rangle&=\left\langle\sum_{i=m-k+1}^{\infty}B_i(r)e_{k+i},\,y\right\rangle\\&=\left\langle\sum_{i=m-k+1}^{\infty}B_i(r)e_{k+i},\,\sum_{i=m+1}^{\infty}y_ie_i\right\rangle\\&=\left\langle(I+S(r))^{-1}(S(r)-S)e_k,\,\sum_{i=m+1}^{\infty}y_ie_i\right\rangle,\end{aligned}$$

and thus

$$\begin{aligned}&\left\langle(I+S(r))^{-1}(S(r)-S)e_k,\ y\right\rangle=\\&\quad=\left\langle\sum_{i=0}^{m-k}B_i(r)Z^ie_k,\ y\right\rangle+\left\langle(I+S(r))^{-1}(S(r)-S)e_k,\ \sum_{i=m+1}^{\infty}y_ie_i\right\rangle\end{aligned}$$

tends to zero as r tends to 1 since, by (2.9),

$$\left|\left\langle(I+S(r))^{-1}(S(r)-S)e_k,\ \sum_{i=m+1}^{\infty}y_ie_i\right\rangle\right|\leqq C\left(\sum_{i=m+1}^{\infty}|y_i|^2\right)^{\frac{1}{2}},$$

and

$$\lim_{r\to 1}\left\langle\sum_{i=0}^{m}B_i(r)e_k,\ y\right\rangle=0. \quad \square$$

E x a m p l e 2.8. The operator $\Phi=(I-Z)^{-1}$ with dom $\Phi=\{(I-Z)\ell_2(\mathbb{Z})\}$ belongs to the Carathéodory class and meets the conditions of the previous theorem.

D i s c u s s i o n. The operator $(I+\Phi)^{-1}=(2I-Z)^{-1}(I-Z)$ belongs to $\mathscr{U}$, and $(I+\Phi)^{-1}(0)=I/2$. The intersection dom $\Phi\cap$ dom Φ^* is dense since dom $\Phi=$ dom Φ^* (as is seen from $(I-Z)x=(I-Z^*)(-Z)x$). Moreover,

$$\langle(\Phi+\Phi^*)(I-Z)x,(I-Z)x\rangle=\langle(I-Z^*)x,(I-Z)x\rangle,$$

and Φ has a positive real part. Finally, since

$$S=-I+2(I+\Phi)^{-1}=-Z(2I-Z)^{-1},$$

it follows that $S_n=\frac{1}{2^n}I$ for $n\geqq 1$ and therefore, $\sum_0^{\infty}n||S_n||<\infty$.

3. The state space $\mathscr{L}(\Phi)$. For any upper triangular contraction S the operator $\mathscr{M}_S^\ell(F) = SF$ of multiplication on the left by S is a contraction from the Hilbert space $\mathscr{U}_2$ of upper triangular Hilbert–Schmidt operators into itself. By $\mathscr{H}(S) \subset \mathscr{U}_2$ we denote the operator range

$$\mathscr{H}(S) = \operatorname{ran}\left(I_{\mathscr{U}_2} - \mathscr{M}_S^\ell \mathscr{M}_S^{\ell *}\right)^{\frac{1}{2}}$$

with the lifted norm

$$\left\| \left(I_{\mathscr{U}_2} - \mathscr{M}_S^\ell \mathscr{M}_S^{\ell *}\right)^{\frac{1}{2}} F \right\|_{\mathscr{H}(S)} = \|(I - P)F\|_{\mathscr{U}_2},$$

where P is the orthogonal projection in $\mathscr{U}_2$ onto $\ker\left(I_{\mathscr{U}_2} - \mathscr{M}_S^\ell \mathscr{M}_S^{\ell *}\right)$. Let $\mathscr{D}_2 = \mathscr{D} \cap \mathscr{U}_2$. In [5], it was shown that the formulas

$$\mathbf{A}_\ell(f) = \left(f - f_{[0]}\right) Z^{-1}, \qquad \mathbf{B}_\ell(E) = \left(S - S_{[0]}\right) E Z^{-1} \tag{3.1}$$

$$\mathbf{C}_\ell(f) = f_{[0]}, \qquad \mathbf{D}_\ell(E) = S_{[0]} E \tag{3.2}$$

define a bounded coisometric colligation

$$\mathscr{V}_\ell = \begin{pmatrix} \mathbf{A}_\ell & \mathbf{B}_\ell \\ \mathbf{C}_\ell & \mathbf{D}_\ell \end{pmatrix} : \begin{pmatrix} \mathscr{H}(S) \\ \mathscr{D}_2 \end{pmatrix} \longrightarrow \begin{pmatrix} \mathscr{H}(S) \\ \mathscr{D}_2 \end{pmatrix} \tag{3.3}$$

which is moreover, closely outer–connected: $\mathscr{H}(S) = \overline{\operatorname{span}}\left\{\operatorname{ran}\,(\mathbf{A}_\ell^*)^n \mathbf{C}_\ell^* \,|\, n \geqq 0\right\}$.

Definition 3.1. Let Φ be a Carathéodory operator with associated sequence of diagonal operators $\widehat{\Phi}_0, \widehat{\Phi}_1, \ldots$. The space $\mathscr{L}(\Phi)$ is the linear space of sequences of diagonal operators $\mathbf{F} = \{F_0, F_1, \ldots\}$ for which the Zadeh extension is of the form

$$\mathbf{F}(r) = \frac{1}{\sqrt{2}}(I + \Psi(r)) f(r), \quad r \in [0, \varepsilon) \tag{3.4}$$

for some $f \in \mathscr{H}(S)$ and some $\varepsilon > 0$ (which may depend on F), equipped with the norm $\|F\|_{\mathscr{L}(\Phi)} = \|f\|_{\mathscr{H}(S)}$.

In (3.4), $r \in [0, \epsilon)$, where $\epsilon > 0$ may depend on $\mathbf{F}$.

Lemma 3.2. *Let $W \in \mathscr{D}$ have the norm strictly less than* 1. *Then the operator $\mathscr{M}_W^r$ of multiplication on the right by W is a contraction in $\mathscr{L}(\Phi)$.*

This fact for the space $\mathscr{H}(S)$ is proved in [5]. The similar fact here follows from the definition of the norm and from the equality

$$\begin{aligned}\left(\mathscr{M}_W^r \mathbf{F}\right)(r) &= \frac{1}{\sqrt{2}}(I + \Phi(r)) f(r) W \\ &= \frac{1}{\sqrt{2}}(I + \Phi(r))(fW)(r) = \frac{1}{\sqrt{2}}(I + \Phi(r))\left(\mathscr{M}_W^r f\right)(r).\end{aligned}$$

Theorem 3.3. *The formulas*

$$\begin{aligned}&\mathbf{A}\mathbf{F} = \{F_1, F_2, \ldots\}, \qquad \mathbf{C}\mathbf{F} = F_0, \qquad \mathbf{D}E = \Phi_0 E, \\ &\mathbf{B}E = \left\{\Phi_1 E^{(-1)}, \Phi_2 E^{(-2)}, \ldots\right\}, \quad (E^{(-j)} = Z^j E Z^{-j})\end{aligned} \tag{3.5}$$

define a bounded operator colligation

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} \quad : \quad \begin{pmatrix} \mathscr{L}(\Phi) \\ \mathscr{D}_2 \end{pmatrix} \longrightarrow \begin{pmatrix} \mathscr{L}(\Phi) \\ \mathscr{D}_2 \end{pmatrix}.$$

Let $U : \mathscr{H}(S) \to \mathscr{L}(\Phi)$ *denote the unitary mapping* $Uf = \mathbf{F}$ *and let* $V : \mathscr{D}_2 \to \mathscr{D}_2$ *denote the operator of multiplication on the left by* $\frac{1}{\sqrt{2}}(I + \Phi_0)$. *The operators* $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ *and* $\mathbf{D}$ *are related to the operators in the colligation* (3.1)–(3.2) *by the equations*

$$\mathbf{A} = U\mathbf{A}_\ell U^* - \frac{1}{\sqrt{2}} U\mathbf{B}_\ell V\mathbf{C}_\ell U^*, \qquad \mathbf{B} = -U\mathbf{B}_\ell V, \tag{3.6}$$

$$\mathbf{C} = V\mathbf{C}_\ell U^*, \qquad \mathbf{D} = I - \sqrt{2}V\mathbf{D}. \tag{3.7}$$

Moreover,

$$\mathbf{A}\mathbf{A}^* = I_{\mathscr{L}(\Phi)}, \quad \mathbf{B} = \mathbf{A}\mathbf{C}^* \quad \text{and} \quad \operatorname{Re}\mathbf{D} = \frac{1}{2}\mathbf{C}\mathbf{C}^*. \tag{3.8}$$

Proof. To prove the first equation in (3.6), we note that in view of (2.7) and definitions of $\mathbf{B}_\ell$ and $\mathbf{C}_\ell$,

$$\begin{aligned}(\Psi(r) - \Psi(0))\, f(0)(rZ)^{-1} &= \tfrac{1}{2}(I + \Psi(r))(S(0) - S(r))(I + \Phi_0)\, f(0)(rZ)^{-1} \\ &= -(U\mathbf{B}_\ell V\mathbf{C}_\ell f)(r).\end{aligned} \tag{3.9}$$

Therefore,

$$\begin{aligned}(\mathbf{AF})(r) &= (\mathbf{F}(r) - F_0)(rZ)^{-1} \\ &= \frac{1}{\sqrt{2}}\left((I + \Psi(r))\, f(r) - (I + \Psi(0))\, f(0)\right)(rZ)^{-1} \\ &= \frac{1}{\sqrt{2}}(I + \Psi(r))((f(r) - f(0))(rZ)^{-1}) + \\ &\quad + \frac{1}{\sqrt{2}}(\Psi(r) - \Psi(0))\, f(0)(rZ)^{-1} \\ &= \left(U\left(\mathbf{A}_\ell - \frac{1}{\sqrt{2}}\mathbf{B}_\ell V\mathbf{C}_\ell\right)U^*\mathbf{F}\right)(r).\end{aligned}$$

To check the second equation in (3.6), we recall that $S(0) = 0$ and write:

$$\begin{aligned}(\mathbf{B}E)(r) &= \sum_{n=1}^{\infty} \Phi_n r^n E^{(-n)} Z^n = \sum_{n=1}^{\infty} \Phi_n r^n Z^n E = (\Psi(r) - \Psi(0))\, E(rZ)^{-1} \\ &= -(U\mathbf{B}_\ell VE)(r),\end{aligned}$$

in view of (3.9). The formulas in (3.7) are readily checked. The proofs of the identities in (3.8) relies on the relations (3.6) and (3.7) and the fact that the colligation $\mathscr{V}_\ell$ is coisometric. For instance, we have:

$$\begin{aligned}\mathbf{A}\mathbf{C}^* &= U\left(\mathbf{A}_\ell\mathbf{C}_\ell^* - \frac{1}{\sqrt{2}}\mathbf{B}_\ell V\mathbf{C}_\ell\mathbf{C}_\ell^*\right)V^* \\ &= -U\mathbf{B}_\ell\left(\mathbf{D}_\ell^* V^* + \frac{1}{\sqrt{2}}V(I_{\mathscr{D}_2} - \mathbf{D}_\ell\mathbf{D}_\ell^*)V^*\right),\end{aligned}$$

and since

$$\begin{aligned}
&\left(\mathbf{D}_\ell V^* + \frac{1}{\sqrt{2}}(I_{\mathscr{D}_2} - \mathbf{D}_\ell \mathbf{D}_\ell^*)V^*\right)(E)\\
&= \frac{1}{\sqrt{2}}\left(S_0^*(I+\Psi_0^*) + \frac{1}{2}(I+\Phi_0)(I - S_0S_0^*)(I+\Phi_0^*)\right)(E)\\
&= \frac{1}{\sqrt{2}}(I+\Phi_0)(E) = VE
\end{aligned}$$

for every $E \in \mathscr{D}_2$, it follows that $\mathbf{AC}^* = -U\mathbf{B}_\ell V = \mathbf{B}$. $\square$

4. Herglotz' formula. In [2] was introduced a $\mathscr{D}$–valued function associated to a bounded upper triangular operator by the formula

$$F^{\triangle}(W) = \sum_0^\infty F_n W^{\{n\}}, \tag{4.1}$$

where $W^{\{n\}} = Z^n(Z^*W)^n$ is in $\mathscr{D}$ when $W \in \mathscr{D}$ and where $F_0, F_1, \ldots$ is the sequence of diagonals associated to F as in (2.1). Formula (4.1) still makes sense for certain sequences of diagonal operators which do not necessarily define a bounded upper triangular operator.

Theorem 4.1. *Let $\Phi \in \mathscr{U}$ be a Carathéodory operator with associated diagonal sequences $\widehat{\Phi}_0, \widehat{\Phi}_1, \ldots$, and assume that the main diagonal of $(I+\Phi)^{-1}$ is equal to the identity. For any $W \in \mathscr{D}$ such that $\|W\| < 1$ and any $E \in \mathscr{D}_2$,*

$$\sum_0^\infty \widehat{\Phi}_n E^{(-n)} W^{\{n\}} = \left(i\,\mathrm{Im}\,\mathbf{D} + \tfrac{1}{2}\mathbf{C}\left(I_{\mathscr{L}(\Phi)} + \mathscr{M}_W^r\mathbf{A}\right)\left(I_{\mathscr{L}(\Phi)} - \mathscr{M}_W^r\mathbf{A}\right)^{-1}\mathbf{C}^*\right)(E), \tag{4.2}$$

where the operators $\mathbf{A}$, $\mathbf{C}$ and $\mathbf{D}$ are defined by (3.5). *The main operator $\mathbf{A}$ is coisometric,* $\mathrm{Re}\,\mathbf{D} = 1/2\mathbf{CC}^*$ *and moreover,*

$$\mathscr{L}(\Phi) = \overline{\mathrm{span}}\,\{\mathrm{ran}\,(\mathbf{A}^*)^n\mathbf{C}^* | n \geqq 0\}\,.$$

Note that if Φ is bounded, the left hand side of (4.2) is equal to $(\Phi E)^{\triangle}(W)$ and that we get back the results of [4]. When one considers the setting of upper triangular Toeplitz operators and takes scalar W, one gets back to the classical Herglotz representation theorem. This is easily seen by using the spectral theorem for (a unitary extension of) the operator $\mathbf{A}$.

P r o o f o f t h e t h e o r e m. For $F \in \mathscr{L}(\Phi)$ we have

$$\left(\mathscr{M}_W^r\mathbf{A}(F)\right)(r) = \left(F(r) - F_{[0]}\right)(rZ)^{-1}W = \sum_{n=1}^\infty r^n Z^n F_{[n]} Z^* W,$$

and so

$$\mathbf{C}\mathscr{M}_W^r\mathbf{A}(F) = F_1 W.$$

In the same manner we get that for $n \geqq 1$,

$$\mathbf{C}\left(\mathscr{M}_W^r\mathbf{A}\right)^n(F) = F_n W^{\{n\}}.$$

Therefore

$$F^{\triangle}(W) = \mathbf{C}\left(I_{\mathscr{L}(\Phi)} - \mathscr{M}_W^r\mathbf{A}\right)^{-1}(F), \tag{4.3}$$

where the inverse exists because $\mathbf{A}$ is a coisometry and $\mathscr{M}_W^r$ is a strict contraction by Lemma 3.2. For all $F \in \mathscr{U}$ and $W \in \mathscr{D}$ with $\|W\| < 1$, it holds that

$$(FZ)^{\triangle}(W) = (FW)^{\triangle}(W) = F^{\triangle}(W^{(-1)})W, \tag{4.4}$$

and thus,

$$(FZ)^{\triangle}(W) = \mathbf{C}\left(I_{\mathscr{L}(\Phi)} - \mathscr{M}_W^r\mathbf{A}\right)^{-1}(F)W.$$

Setting $F = \mathbf{B}(E)$, we conclude that for all $E \in \mathscr{D}_2$ and $W \in \mathscr{D}$ with $\|W\| < 1$,

$$\sum_{n=1}^{\infty} \Phi_n E^{(-n)} W^{\{n\}} = \mathscr{M}_W^r\mathbf{C}\big(I_{\mathscr{L}(\Phi)} - \mathscr{M}_{W^{(-1)}}^r\mathbf{A}\big)^{-1}\mathbf{B}(E).$$

In view of (3.8) and since

$$\mathbf{D}(E) = \Phi_0 E, \quad \mathscr{M}_{W^{(-1)}}^r\mathbf{A} = \mathbf{A}\mathscr{M}_W^r, \quad \mathscr{M}_W^r\mathbf{C} = \mathbf{C}\mathscr{M}_W^r,$$

it follows that

$$\begin{aligned}
\sum_{n=0}^{\infty} \Phi_n E^{(-n)} W^{\{n\}} &= \Big(\mathbf{D} + \mathscr{M}_W^r\mathbf{C}\big(I_{\mathscr{L}(\Phi)} - \mathscr{M}_{W^{(-1)}}^r\mathbf{A}\big)^{-1}\mathbf{B}\Big)(E)\\
&= \Big(\mathbf{D} + \mathbf{C}\mathscr{M}_W^r\left(I_{\mathscr{L}(\Phi)} - \mathbf{A}_W^r\right)^{-1}\mathbf{B}\Big)(E)\\
&= i\,\mathrm{Im}\,\mathbf{D} + \Big(\tfrac{1}{2}\mathbf{C}\mathbf{C}^* + \mathbf{C}\left(I_{\mathscr{L}(\Phi)} - \mathscr{M}_W^r\mathbf{A}\right)^{-1}\mathscr{M}_W^r\mathbf{A}\mathbf{C}^*\Big)(E)\\
&= i\,\mathrm{Im}\,\mathbf{D} + \Big(\tfrac{1}{2}\mathbf{C}\left(I_{\mathscr{L}(\Phi)} + \mathscr{M}_W^r\mathbf{A}\right)\left(I_{\mathscr{L}(\Phi)} - \mathscr{M}_W^r\mathbf{A}\right)^{-1}\mathbf{C}^*\Big)(E).
\end{aligned}$$

Finally, we prove that $\mathscr{V}$ is closely outer–connected. Assume that $F \in \mathscr{L}(\Phi)$ is orthogonal to all operators of the form $\left(I_{\mathscr{L}(\Phi)} - \mu^*\mathbf{A}^*\right)^{-1}\mathbf{C}^*(E)$ with $E \in \mathscr{D}_2$ and $\mu \in \mathbb{D}$. Then

$$\begin{aligned}
0 = \Big\langle F, \left(I_{\mathscr{L}(\Phi)} - \mu^*\mathbf{A}^*\right)^{-1}\mathbf{C}^*(E)\Big\rangle_{\mathscr{L}(\Phi)} &= \Big\langle \mathbf{C}\left(I_{\mathscr{L}(\Phi)} - \mu\mathbf{A}\right)^{-1}(F), E\Big\rangle_{\mathscr{D}_2}\\
&= \left\langle F^{\triangle}(\mu I), E\right\rangle_{\mathscr{D}_2}
\end{aligned}$$

which implies that $F^{\triangle}(\mu I) = 0$ for all $\mu \in \mathbb{D}$ and thus, $F = 0$. $\square$

References

[1] D. Alpay, V. Bolotnikov, P. Dewilde et A. Dijksma, Sections de Brune en théorie des systèmes non stationnaires. C. R. Acad. Sci. Paris Sér. I Math. **330**, 173–178 (2000).

[2] D. Alpay and P. Dewilde. Time-varying signal approximation and estimation. In: Signal processing, scattering and operator theory, and numerical methods (Amsterdam, 1989). M. Kaashoek, J. H. van Schuppen and A. C. M. Ran, eds.. Prog. Syst. Control Theory, **5**, 1–22. Boston 1990.

[3] D. Alpay, P. Dewilde and H. Dym, Lossless inverse scattering and reproducing kernels for upper triangular operators. In: Extension and interpolation of linear operators and matrix functions, 61–135, Basel 1990.

[4] D. Alpay, A. Dijksma and Y. Peretz, Nonstationary analogs of the Herglotz representation theorem: the discrete case. J. Funct. Anal. **166**(1), 85–129 (1999).

[5] D. Alpay and Y. Peretz, Realizations for Schur upper triangular operators. In: Contributions to operator theory in spaces with an indefinite metric, A. Dijksma, I. Gohberg, M. Kaashoek and R. Mennicken, eds.. Oper. Theory: Adv. Appl. **106**, 37–90 (1998).

[6] M. S. Brodskiĭ, Triangular and Jordan representations of linear operators. Amer. Math. Soc., Providence, R.I. 1971. Translated from the Russian by J. M. Danskin, Transl. Math. Monographs **32**.

[7] E. DEPRETTERE and P. DEWILDE, The generalized Schur algorithm. Oper. Theory: Adv. Appl. **29**, 97–115 (1988).

[8] P. DEWILDE and A.-J. VAN DER VEEN, Time-varying systems and computations. Boston, MA 1998.

[9] H. DYM and B. FREYDIN, Bitangential interpolation for upper triangular operators. In: Topics in interpolation theory, H. Dym, B. Fritzsche, V. Katsnelson and B. Kirstein, eds.. Oper. Theory: Adv. Appl. **95**, 105–142 (1997).

[10] C. FOIAS, A. E. FRAZHO, I. GOHBERG and M. A. KAASHOEK, Discrete time–variant interpolation as classical interpolation with an operator argument. Integral Equations Operator Theory **26**, 371–403 (1996).

[11] M. A. PETERS and P. A. IGLESIAS, Minimum entropy control for time-varying systems. Boston, MA 1997.

Eingegangen am 21. 8. 2000

Anschrift der Autoren:

Daniel Alpay, Boris Freydin
Department of Mathematics
Ben–Gurion University of the Negev
Beer-Sheva 84105, Israel

Aad Dijksma
Department of Mathematics
University of Groningen
9700 AV Groningen
The Netherlands

Vladimir Bolotnikov
Department of Mathematics
The College of William and Mary
Williamsburg, VA 23187-8795, USA